A Note On Gorenstein Projective Conjecture II

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Abstract In this paper, we prove that Gorenstein projective conjecture is left and right symmetric and the co-homology vanishing condition can not be reduced in general. We also prove that algebras of finite left or right finistic dimension satisfy Gorenstein projective conjecture. As a result, Gorenstein projective conjecture (generalized Nakayama conjecture) is true for Gorenstein algebra. Moreover, Gorenstein projective conjecture is proved to be true for CM-finite algebras.

Keywords Gorenstein projective, CM-finite algebras, Gorenstein projective conjecture
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1 Introduction

For the representation theory of Artinian algebras, the generalized Nakayama conjecture (GNC) is everything. It was proposed by Auslander and Reiten, which says that M is projective if $\mathrm{Ext}^i_\Lambda(M\bigoplus\Lambda, M\bigoplus\Lambda)=0$ for any $i\geq 1$(See [2,3]). Achievements for special cases have been got by K. R. Fuller, B. Zimmermann-Huisgen, A. Maròti and G. Wilson...(See [10,15,17]). In general it is still open now. As a special case of generalized Nakayama conjecture, Luo and Huang proposed the following Gorenstein projective conjecture (GPC) in 2008:

Let Λ be an Artinian algebra and let M be a Gorenstein projective module. Then M is projective if and only if $\mathrm{Ext}^i_\Lambda(M,M)=0$ for any $i\geq 1$(See [14,19]).

It is still unknown whether the generalized Nakayama conjecture is left and right symmetric. But as we stated Gorenstein projective conjecture is a special case of generalized Nakayama conjecture. So what about the left and right symmetric property of Gorenstein projective conjecture? In this paper, we will give a positive answer to this question.

By the definition of Gorenstein projective conjecture, for an algebra Λ the truth of generalized Nakayama conjecture implies the truth of Gorenstein projective conjecture. So we can get a large class of algebras satisfying Gorenstein projective conjecture. It is interesting

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to ask: Is there an algebra satisfying Gorenstein projective conjecture while for which the generalized Nakayama conjecture is unknown?

Recall that an algebra is called CM-finite (of finite Cohen-Macaulay type) if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules. CM-finite algebras are studied by several authors recently (see [5,6,7,12,13]). Although the generalized Nakayama conjecture for this class of algebras is unknown, we will give a positive answer to the second question above.

The paper is organized as follows:

In Section 2, based on some facts of Gorenstein projective modules, we will show the symmetric property of Gorenstein projective conjecture. Moreover, an example is given to show that the condition '$\mathrm{Ext}^i_\Lambda(M,M)=0$ for any $i\geq 1$' in Gorenstein projective conjecture can not be reduced to '$\mathrm{Ext}^i_\Lambda(M,M)=0$ for some positive integer t and any $1\leq i\leq t$.'

In Section 3, we will show algebras with finite left or right finistic dimension satisfy Gorenstein projective conjecture. This ensures the existence of a large class of algebras satisfying Gorenstein projective conjecture.

In Section 4, we will prove that CM-finite algebras satisfy the Gorenstein projective conjecture by showing the Gorenstein projective conjecture holds for algebras with finite self-orthogonal indecomposable Gorenstein projective modules (up to isomorphisms).

Throughout the paper, Λ is an Artinian algebra and all modules are finitely generated left Λ-modules.

2 Symmetric property of Gorenstein projective conjecture

In this section we will show the symmetric property of Gorenstein projective conjecture. First, we need to recall some notions and lemmas. The following definition is due to Auslander, Briger, Enochs and Jenda (see [1,8,9]).

Definition 2.1 A module M is called *Gorenstein projective* if for any $i\geq 1$

(1) $\mathrm{Ext}^i_\Lambda(M,\Lambda)=0$

(2) $\mathrm{Ext}^i_\Lambda(\mathrm{Tr}\,M,\Lambda)=0$

Where $\mathrm{Tr}\,M$ denotes the Auslander transpose of M.

Let $\cdots\to P_2(M)\to P_1(M)\to M\to 0$ be a minimal projective resolution of M. Denoted by Ω^iM the i-th syzygy of M. Dually, one can define $\Omega^{-i}M$. We remark that for any $i\geq 0$ Ω^iM is a Gorenstein projective if so is M. Let $\mathscr{C}$ be the subcategory of $\mathrm{mod}\,\Lambda$ consisting of modules M such that $\mathrm{Ext}^j_\Lambda(M,\Lambda)=0$ for any $j\geq 1$ and $\mathscr{D}$ a subcategory of $\mathrm{mod}\,\Lambda$ consisting of Gorenstein projective modules. We use $\underline{\mathscr{C}}$ and $\underline{\mathscr{D}}$ to denote the stable subcategory of $\mathscr{C}$

and $\mathscr{D}$ modulo projectives, respectively. We recall the following proposition from [1].

Proposition 2.2 (1) $\Omega : \underline{\mathscr{C}} \to \underline{\mathscr{C}}$ *is a fully faithful functor .*

(2) $\Omega : \underline{\mathscr{D}} \to \underline{\mathscr{D}}$ *is an equivalence.*

(3) $(-)^* = \mathrm{Hom}(-, \Lambda) : \mathscr{D} \to \mathscr{D}^o$ *is a duality, where* $\mathscr{D}^o$ *denotes the subcategory of Gorenstein projective right* Λ*-modules.*

Proof. (1) is a result of Auslander and Bridger. One can get (2) by (1) and the remark above. (3) is well-known. □

Recall that a module M is called self-orthogonal if $\mathrm{Ext}^j_\Lambda(M, M) = 0$ for any $j \geq 1$. The following self-orthogonal property is essential to the main result in this section.

Lemma 2.3 *Let* Λ *be an algebra. Then for any* $M \in \mathscr{D}$ *and* $i \geq 1$, M *is self-orthogonal if and only if* M^* *is self-orthogonal.*

Proof. Since $(-)^*$ is a duality between $\mathscr{D}$ and $\mathscr{D}^o$, it is enough to show that $\mathrm{Ext}^i_\Lambda(M, M) = 0$ implies $\mathrm{Ext}^i_\Lambda(M^*, M^*) = 0$.

One can take the following minimal projective resolution of M:

$$\cdots \to P_1 \to P_0 \to M \to 0 \qquad (1)$$

Applying the functor $\mathrm{Hom}(-, M)$ to sequence (1) above, since $\mathrm{Ext}^i_\Lambda(M, M) = 0$ we get the following exact sequence

$$0 \to \mathrm{Hom}(M, M) \to \mathrm{Hom}(P_0, M) \to \mathrm{Hom}(P_1, M) \to \cdots \qquad (2)$$

On the other hand, applying the functor $(-)^*$ to the sequence (1), since $M \in \mathscr{D} \subseteq \mathscr{C}$ one can show the following exact sequence

$$0 \to M^* \to {P_0}^* \to {P_1}^* \to \cdots \qquad (3)$$

Then by using the functor $\mathrm{Hom}(M^*, -)$ on the exact sequences (3), one has the following exact sequence

$$0 \to \mathrm{Hom}(M^*, M^*) \to \mathrm{Hom}(M^*, {P_0}^*) \to \mathrm{Hom}(M^*, {P_1}^*) \to \cdots \qquad (4)$$

Using Proposition 2.2(3), we get $\mathrm{Ext}^i_\Lambda(M^*, M^*) \simeq \mathrm{Ext}^i_\Lambda(M, M) = 0$ by comparing sequences (2) with (4). □

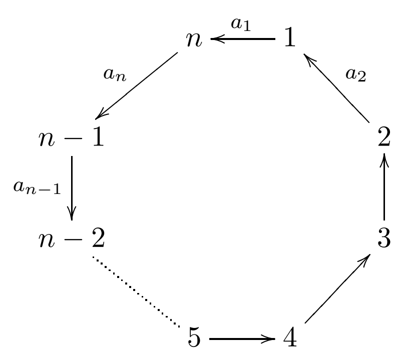
Although the symmetric property of Generalized Nakayama conjecture is still unknown, we are able to show the symmetric properties of Gorenstein projective conjecture.

Theorem 2.4 *Let Λ be an algebra and let Λ^o be the opposite ring of Λ. Then Λ satisfies the Gorenstein projective conjecture if and only if Λ^o satisfies the Gorenstein projective conjecture.*

Proof. $\Rightarrow$ Assume that $N \in \mathscr{D}^o$ and $\mathrm{Ext}^i_\Lambda(N, N) = 0$ for any $i \geq 1$. By Proposition 2.2, there is a $M \in \mathscr{D}$ such that $M^* \simeq N$. By Lemma 2.3 one gets that $\mathrm{Ext}^i_\Lambda(M, M) = 0$. Note that Λ satisfies the Gorenstein projective conjecture, we have M is projective, and hence $N \simeq M^*$ is projective. Conversely, one can formula the proof above. $\square$

Notice that Gorenstein projective conjecture is a special case of Generalized Nakayama conjecture. It is natural to consider whether the assumption of Gorenstein projective conjecture can be reduced. In particular, whether can the condition '$\mathrm{Ext}^i_\Lambda(M, M) = 0$ for any $i \geq 1$' in GPC be reduced to '$\mathrm{Ext}^i_\Lambda(M, M) = 0$ for some positive integer t and any $1 \leq i \leq t$? At the end of this section, we construct an example to give a negative answer to the question.

Example 2.5 Let $n > t + 1$ be a positive integer and let Λ be the algebra generated by the following quiver



modulo the ideal $\{a_n a_1 = 0, a_i a_{i+1} = 0 | 1 \leq i \leq n - 1\}$. Denoted by $S(j)$ the simple module according to the dot j. Then

(1) Λ is a Nakayama self-injective algebra.

(2) $S(j)$ is Gorenstein projective such that $\mathrm{Ext}^i_\Lambda(S(j), S(j)) = 0$ for $t \geq i \geq 1$ and $1 \leq j \leq n$, but it is not projective.

3 Generalized Nakayama conjecture and finistic dimension

In this section we will show that Gorenstein projective conjecture holds for a large class of algebras in terms of finistic dimension and generalized Nakayama conjecture. Firstly, we need to recall the following equivalent version of generalized Nakayama conjecture (see [2] or [18]).

GNC': Let Λ be an Artin algebra and let S be a simple module Λ-module. Then there exists a non-negative integer n such that $\mathrm{Ext}_\Lambda^n(S,\Lambda) \neq 0$.

Denote by $lfd\Lambda = Sup\{\mathrm{pd}_\Lambda M < \infty | M \in \mathrm{mod}\,\Lambda\}$ the left finistic dimension of Λ (see [4]). Dually, one can define the right finistic dimension of Λ. We use $\mathrm{mod}\,\Lambda^o$ to denote the category of finitely generated right Λ-modules. Denote by $\mathrm{pd}_\Lambda M$ and $\mathrm{id}_\Lambda M$ projective and injective dimension of M, respectively. We have the following:

Theorem 3.1 *Let Λ be an algebra with $lfd\Lambda = t < \infty$ for some non-negative integer t. Then*

(1) for any non-zero simple module $S \in \mathrm{mod}\,\Lambda^o$, there is a non-negative integer i such that $\mathrm{Ext}_\Lambda^i(S,\Lambda) \neq 0$, that is, Λ^o satisfies generalized Nakayama conjecture.

(2) Λ satisfies Gorenstein projective conjecture.

Proof. (1) If $\mathrm{pd}_{\Lambda^o} S = j \leq t$, then it is not difficult to show that $\mathrm{Ext}_\Lambda^j(S,\Lambda) \neq 0$. Now we only have to show the case of $\mathrm{pd}_{\Lambda^o} S = \infty$.

On the contrary, suppose that $\mathrm{Ext}_\Lambda^i(S,\Lambda) = 0$ for any $i \geq 0$. Taking the following minimal projective resolution of S:

$\cdots \to P_{t+1}(S) \to P_t(S) \to \cdots \to P_1(S) \to P_0(S) \to S \to 0$ and applying the functor $(-)^* = \mathrm{Hom}_\Lambda(-,\Lambda)$, one can get the following exact sequence:

$0 \to P_0(S)^* \to P_1(S)^* \to \cdots \to P_t(S)^* \to P_{t+1}(S)^* \to \mathrm{Tr}\Omega^{t+1}S \to 0 \quad (*)$, where we use Ω^i and Tr to denote the i-th syzygy functor and the Auslander-Bridger transpose, respectively.

Since $\mathrm{pd}_{\Lambda^o} S = \infty$, one gets that $\Omega^{t+1}S$ is not projective and hence $\mathrm{Tr}\Omega^{t+1}S$ is not zero. Then by [16, Proposition 4.2]$(*)$ is a minimal projective resolution of $\mathrm{Tr}\Omega^{t+1}S$. So we get $\mathrm{pd}_\Lambda \mathrm{Tr}\Omega^{t+1}S = t+1 > t$, a contradiction.

(2) Since Gorenstein projective conjecture is a special case of Generalized Nakayama conjecture, one can show Λ^o satisfies Gorenstein projective conjecture by (1). Then by Theorem 2.4 one gets the assertion. $\square$

The following are straight results of Theorem 3.1 which generalize the result of Huang and Luo in [14].

Corollary 3.2 *(1) If Λ be left Gorenstein, i.e.* $\mathrm{id}_\Lambda \Lambda = m \geq 0$ *for some integer $m \geq 0$. Then Λ^o satisfies generalized Nakayama conjecture.*

(2) If Λ is commutative Artin or Artin local, then Λ satisfies generalized Nakayama conjecture.

(3) Algebras Λ in both (1) and (2) satisfies Gorenstein projective conjecture.

Proof. (1) By [4, Proposition 4.3] we get that $lfd\Lambda \leq id_\Lambda\Lambda$, and hence the first assertion holds. The second follows from $lfd\Lambda = rfd\Lambda = 0$. It is not difficult to show the third one since Gorenstein projective conjecture is symmetric by Theorem 2.4 and it is a special case of generalized Nakayama conjecture. $\square$

We should remark that the dual version of both Theorem 3.1 and Corollary 3.2(1) also hold true. We also note that the Gorenstein projective conjecture does not necessarily depend on generalized Nakayama conjecture. This will be showed in next section.

4 Gorenstein projective conjecture for CM-finite algebras

In this section we try to find a class of algebras which satisfy Gorenstein projective conjecture and for which the generalized Nakayama conjecture is unknown. They are CM-finite algebras. We begin with the following definition due to Beligiannis

Definition 4.1 An algebra is called CM-finite (of finite Cohen-Macaulay type) if there are only finite number of indecomposable Gorenstein projective modules (up to isomorphisms).

Remark 4.2 (1) Algebras of finite representation type or finite global dimension are CM-finite.

(2) There does exist a CM-finite algebra Λ such that Λ is of infinite type and the global dimension of Λ is infinite [13].

(3) There does exist a CM-finite algebra which is not Gorenstein [5].

(4)An algebra with a trivial maximal n-orthogonal subcategory for some positive integer n is CM-finite [11].

Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{D}^o$ be as in Section 2. The following lemma partly from [1] plays an important role in the proof of the main results.

Lemma 4.3 *For any $M \in \mathscr{C}$ and $N \in \operatorname{mod}\Lambda$, then* $\operatorname{Ext}^1_\Lambda(M,N) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^1 M, N)$ *and hence* $\operatorname{Ext}^i_\Lambda(M,N) \simeq \underline{\operatorname{Hom}}_\Lambda(\Omega^i M, N)$ *for any* $i \geq 1$.

Proof. The first assertion is a result of Auslander and Bridger. For the second one, the case $i = 1$ is clear. We only need to show the case $i \geq 2$. Taking a minimal projective resolution of M, one gets $\operatorname{Ext}^i_\Lambda(M,N) \simeq \operatorname{Ext}^1_\Lambda(\Omega^{i-1}M,N)$ for any $i \geq 2$. Notice that $M \in \mathscr{C}$, by Proposition 2.2 one can show $\Omega^{i-1}M \in \mathscr{C}$. Using the first assertion, we are done. $\square$.

The following proposition gives a connection between the self-orthogonal property of M and that of $\Omega^i M$ for any $i \geq 0$.

Proposition 4.4 *Let $M \in \mathscr{C}$ $(\mathscr{D})$. Then*

(1) $\Omega^i M$ is self-orthogonal in $\mathscr{C}$ ($\mathscr{D}$) for any $i \geq 0$ if M is self-orthogonal.

(2) If $M \in \mathscr{D}$ is self-orthogonal, then $\mathrm{Tr}M$ is self-orthogonal in $\mathscr{D}^o$.

Proof. (1) For the case $i = 0$, there is nothing to prove. By Proposition 2.2, we only need to prove the case $i = 1$, that is, $\mathrm{Ext}^j_\Lambda(\Omega M, \Omega M) = 0$ for any $j \geq 1$. One gets $\mathrm{Ext}^j_\Lambda(\Omega M, \Omega M) \simeq \underline{\mathrm{Hom}}_\Lambda(\Omega^{j+1}M, \Omega M) \simeq \underline{\mathrm{Hom}}_\Lambda(\Omega^j M, M)$ by Proposition 2.2 and Lemma 4.3. Using the second equation of Lemma 4.3, one can show $\underline{\mathrm{Hom}}_\Lambda(\Omega^j M, M) \simeq \mathrm{Ext}^j_\Lambda(M, M) = 0$ since M is self-orthogonal.

(2) Taking a minimal projective resolution of M, it is not difficult to show that $\mathrm{Tr}M \simeq (\Omega^2 M)^*$ since $M \in \mathscr{D}$. By Propositions 2.2 and 4.4(1), $\Omega^2 M$ is also self-orthogonal in $\mathscr{D}$. Then by Lemma 2.3 and Proposition 2.2 one gets the assertion. □

The following proposition is crucial to the main results.

Proposition 4.5 *Let Λ be an algebra with only finite (up to isomorphisms) self-orthogonal indecomposable modules in $\mathscr{D}$ ($\mathscr{C}$) and let M be a self-orthogonal indecomposable module in $\mathscr{D}$ ($\mathscr{C}$). Then M is projective.*

Proof. Denoted by $\{M_1, M_2, ..., M_t\}$ the complete set of non-isomorphic self-orthogonal indecomposable modules in $\mathscr{D}$ ($\mathscr{C}$). Then $M \simeq M_i$ for some $1 \leq i \leq t$.

Suppose that M is not projective. Then by Proposition 2.2 and Proposition 3.4, we have the following set of self-orthogonal indecomposable modules $\mathcal{S} = \{\Omega^i M | 1 \leq i\}$ in $\mathscr{D}$ ($\mathscr{C}$).

We claim that there are two modules $\Omega^i M, \Omega^j M$ in $\mathcal{S}$ such that $\Omega^i M \simeq \Omega^j M$ for some $i < j$. Otherwise, one gets infinite number of non-isomorphic self-orthogonal indecomposable modules in $\mathscr{D}$ ($\mathscr{C}$), a contradiction. Again by Proposition 2.2, one gets $M \simeq \Omega^{j-i}M$.

Considering the following exact sequence $0 \to \Omega^{j-i}M \to P \to \Omega^{j-i-1}M \to 0$, we will show $\mathrm{Ext}^1_\Lambda(\Omega^{j-i-1}M, \Omega^{j-i}M) = 0$. Since $\Omega^{j-i-1}M \in \mathscr{D}$ ($\mathscr{C}$) and $M \simeq \Omega^{j-i}M$, we get $\mathrm{Ext}^1_\Lambda(\Omega^{j-i-1}M, \Omega^{j-i}M) \simeq \underline{\mathrm{Hom}}_\Lambda(\Omega^{j-i}M, M) \simeq \mathrm{Ext}^{j-i}_\Lambda(M, M) = 0$ by Proposition 2.2 and Lemma 3.3, and hence M is projective, a contradiction. The assertion holds true. □

Now we are in the position to show the main result of this section.

Theorem 4.6 *Let Λ be CM-finite. Then Λ satisfies Gorenstein projective conjecture.*

Proof. Since Λ is CM-finite, then there are only finite (up to isomorphisms) indecomposable modules in $\mathscr{D}$. One can show the result by Proposition 4.5. □

Although the generalized Nakayama conjecture for CM-finite algebras is unknown now, we give a new description of generalized Nakayama conjecture for CM-finite algebras.

Proposition 4.7 *Let Λ be a CM-finite algebra and let M be a Λ-module satisfying $\mathrm{Ext}^i_\Lambda(M, M \bigoplus \Lambda) = 0$ for any $i \geq 1$. Then the following are equivalent.*

(1) M is projective.

(2) M is Gorenstein projective.

Proof. (1) $\Rightarrow$ (2) is trivial. The converse follows from Theorem 4.6. $\square$

We end this section with two open questions related to this paper.

Question 1 Does the Gorenstein projective conjecture hold for virtually Gorenstein algebras (see [5])?

Question 2 Does the Generalized Nakayama Conjecture hold for CM-finite algebras?

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